Exercise 2.2.3

Show that $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(u, x, t)$ is linear if $Q = \alpha(x, t)u + \beta(x, t)$ and, in addition, homogeneous if $\beta(x, t) = 0$.

Solution

Suppose that $Q = \alpha(x, t)u + \beta(x, t)$. The differential equation becomes

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + \beta(x, t).$$

Isolate $\beta(x, t)$ on the right side.

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} - \alpha(x, t)u = \beta(x, t)$$

Write the left side as an operator L acting on u.

$$\begin{aligned} \left[\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} - \alpha(x, t)\right] u &= \beta(x, t) \\ L(u) &= \beta(x, t) \end{aligned}$$

Notice that the PDE is homogeneous if $\beta(x, t) = 0$. The aim now is to show that L is linear,

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2),$$

where c_1 and c_2 are arbitrary constants and u_1 and u_2 are solutions to the PDE. We have

$$\begin{aligned} L(c_1 u_1 + c_2 u_2) &= \left[\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} - \alpha(x, t)\right] (c_1 u_1 + c_2 u_2) \\ &= \frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) - k\frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2) - \alpha(x, t)(c_1 u_1 + c_2 u_2) \\ &= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - c_1 k\frac{\partial^2 u_1}{\partial x^2} - c_2 k\frac{\partial^2 u_2}{\partial x^2} - c_1\alpha(x, t)u_1 - c_2\alpha(x, t)u_2 \\ &= c_1\frac{\partial u_1}{\partial t} - c_1 k\frac{\partial^2 u_1}{\partial x^2} - c_1\alpha(x, t)u_1 + c_2\frac{\partial u_2}{\partial t} - c_2 k\frac{\partial^2 u_2}{\partial x^2} - c_2\alpha(x, t)u_2 \\ &= c_1\left[\frac{\partial u_1}{\partial t} - k\frac{\partial^2 u_1}{\partial x^2} - \alpha(x, t)u_1\right] + c_2\left[\frac{\partial u_2}{\partial t} - k\frac{\partial^2 u_2}{\partial x^2} - \alpha(x, t)u_2\right] \\ &= c_1\left[\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} - \alpha(x, t)\right] u_1 + c_2\left[\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} - \alpha(x, t)\right] u_2 \\ &= c_1 L(u_1) + c_2 L(u_2). \end{aligned}$$

Therefore, $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(u, x, t)$ is linear if $Q = \alpha(x, t)u + \beta(x, t)$ and, in addition, homogeneous if $\beta(x, t) = 0$.